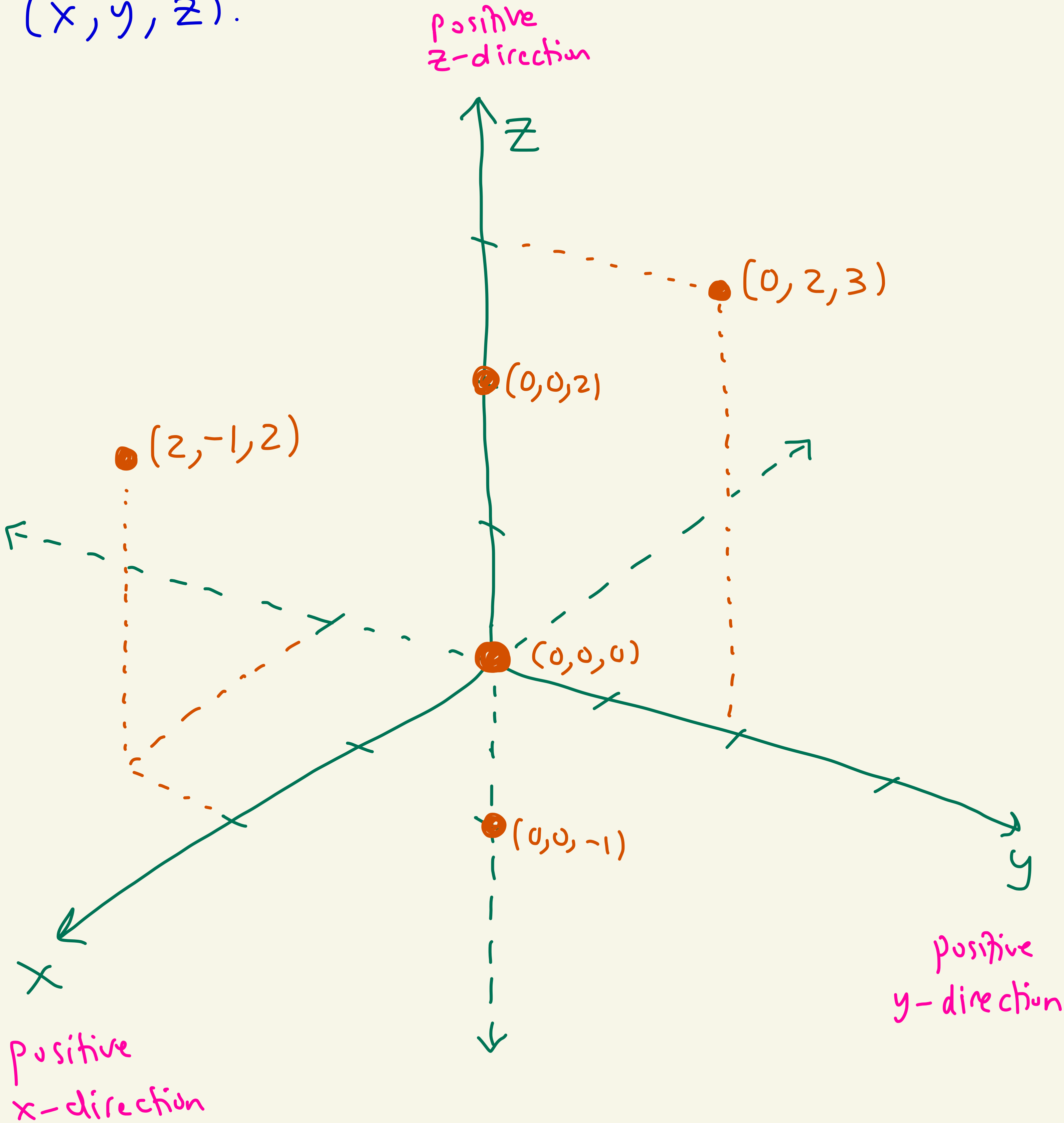
Topic 1 - Functions and surfaces

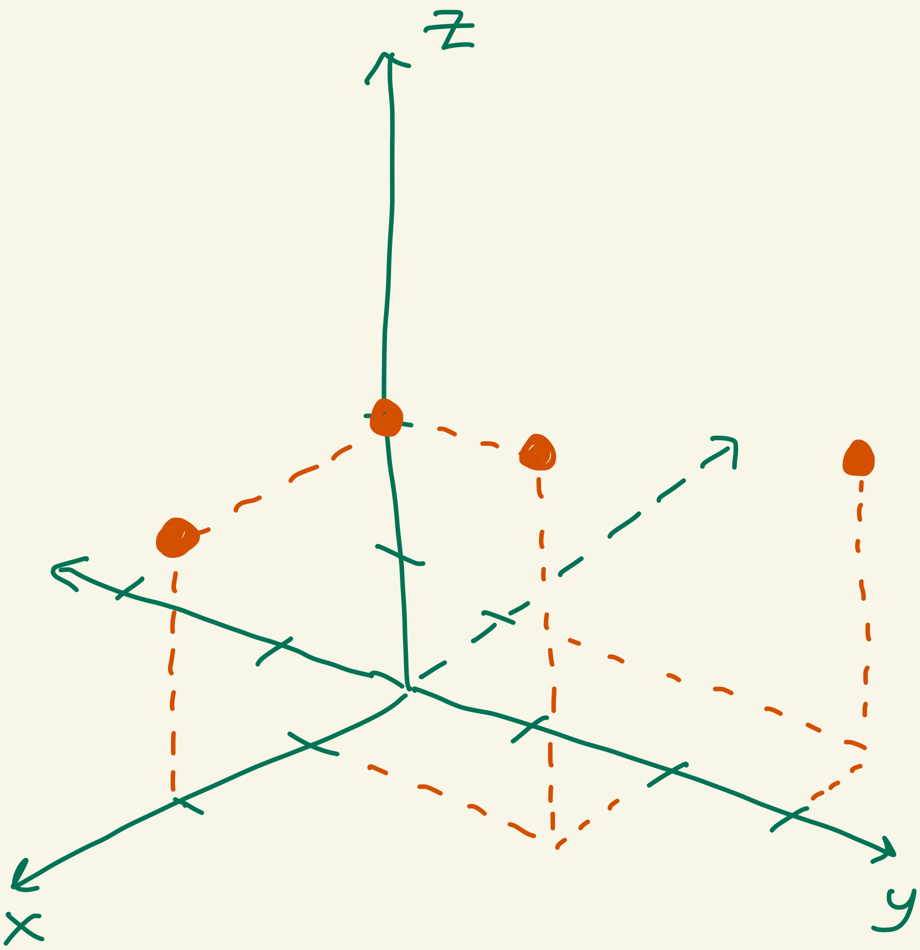
Let's review plotting points in 3-dimensions with coordinates (x, y, z).



<u>Def</u>: A function of two variables f assigns to each (x,y) pair a unique number $f(x,y)$. The <u>domain</u> of f consists of all (x,y) that can be plugged into f.

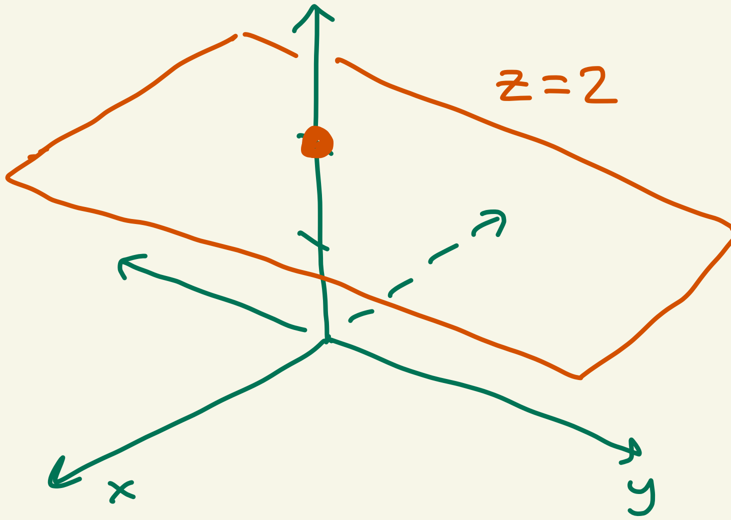
<u>Ex</u>: Let $f(x,y)=2$ for all (x,y).
For plotting purposes, let $z=f(x,y)$.
That is, let $z=2$

x	y	z=2
0	0	2
1	2	2
-1	3	2
2	0	2
⋮	⋮	⋮



The domain of f is all (x,y) since we can plug any (x,y) into $f(x,y)=2$.

If we imagine potting all the x, y values we get the surface $z=2$ which is a plane. The plane is flat and going infinitely in the x and y directions but always at height $z=2$.

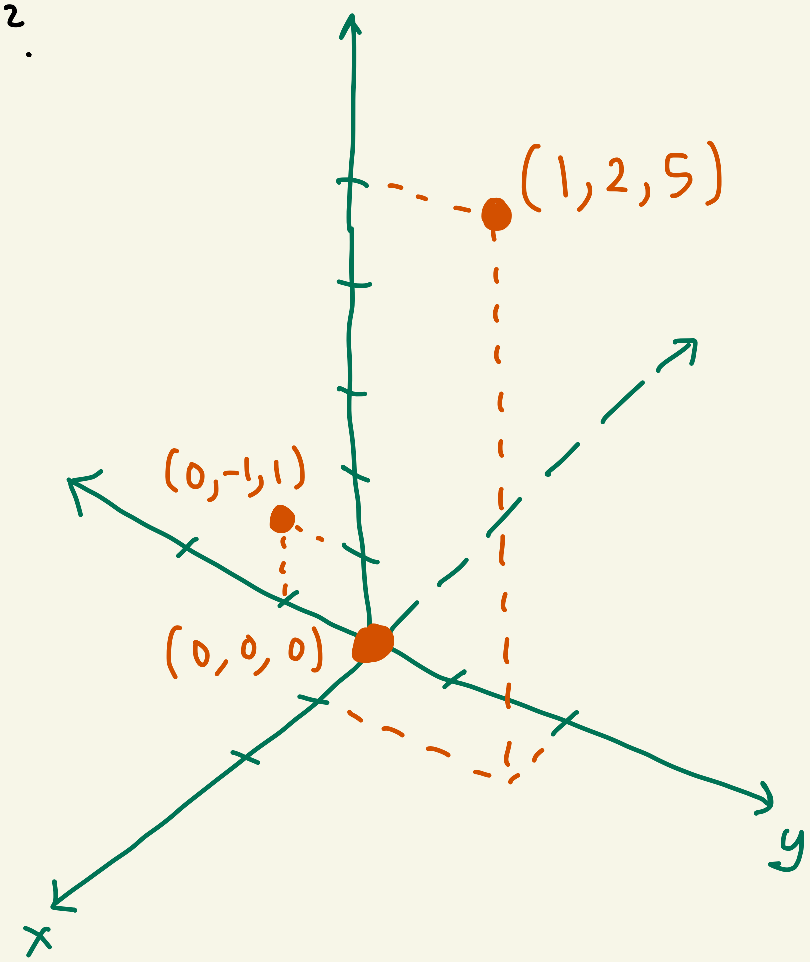


I drew it finitely in size because we can't draw the infinite.

Ex: Let $f(x,y) = x^2 + y^2$.

Let's compute some values.
For plotting purposes, let $z = f(x,y)$.
That is, let $z = x^2 + y^2$.

x	y	$z = x^2 + y^2$
0	0	$0^2 + 0^2 = 0$
1	2	$1^2 + 2^2 = 5$
0	-1	$0^2 + (-1)^2 = 1$
1	4	$1^2 + 4^2 = 17$
$\frac{1}{2}$	π	$\left(\frac{1}{2}\right)^2 + \pi^2 \approx 10.12$
$\vdots$	$\vdots$	$\vdots$



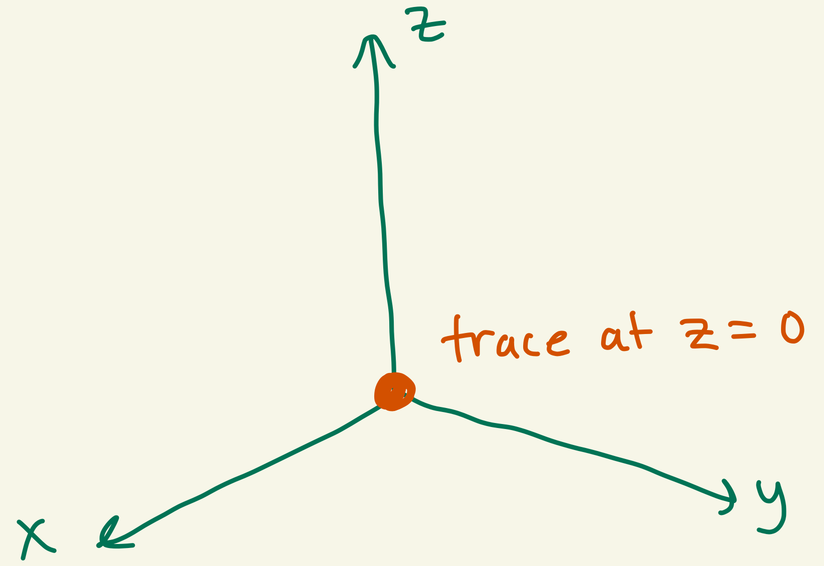
The domain of $f(x,y) = x^2 + y^2$ consists of all (x,y).

Let's try to graph the picture of $z = f(x,y)$, that is $z = x^2 + y^2$, for all (x,y).

We will use "traces" to do this.

$z=0$ trace:

When $z=0$ we have $0=x^2+y^2$.
This only has one solution: $x=0, y=0$.
Thus, the trace at $z=0$ consists of the point $(x,y,z)=(0,0,0)$

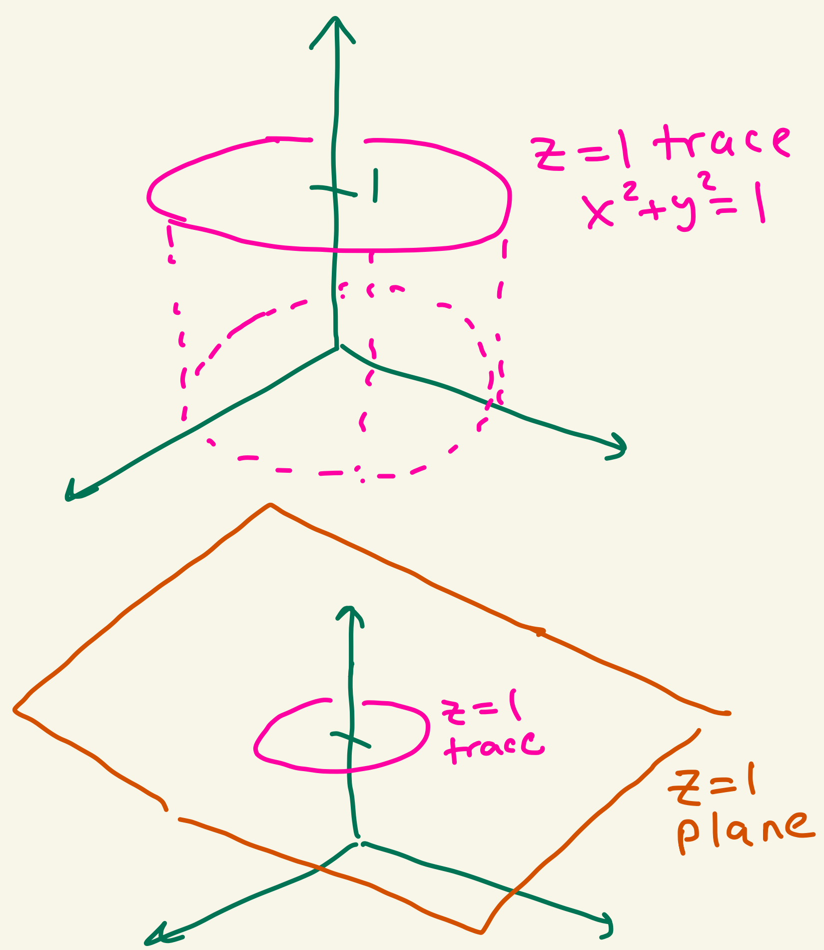


$z=1$ trace:

When $z=1$ we have

$$1=x^2+y^2$$

This is a circle of radius 1 centered at the origin.
But when we draw it we draw it at the height $z=1$.
The trace lives in the $z=1$ plane.

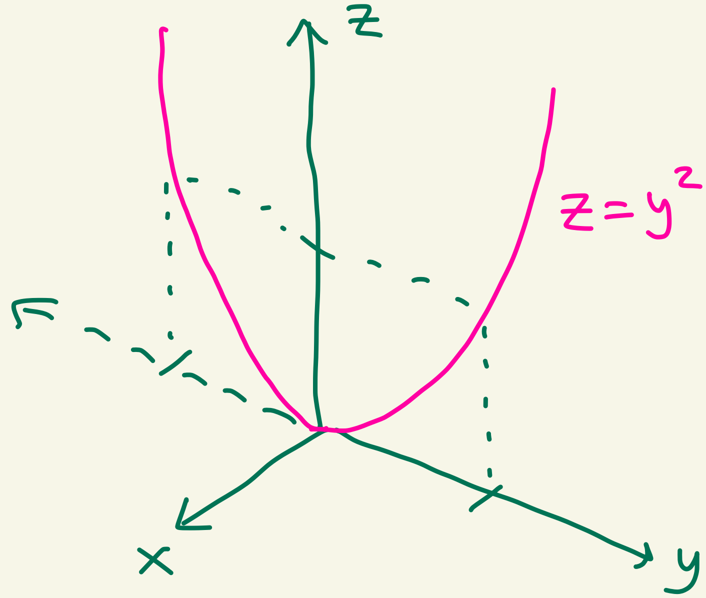


$x=0$ trace:

When $x=0$ we get
$z = x^2+y^2 = 0^2+y^2 = y^2$

So we get $z=y^2$.

The trace lives in the $x=0$ plane (the yz-axis)

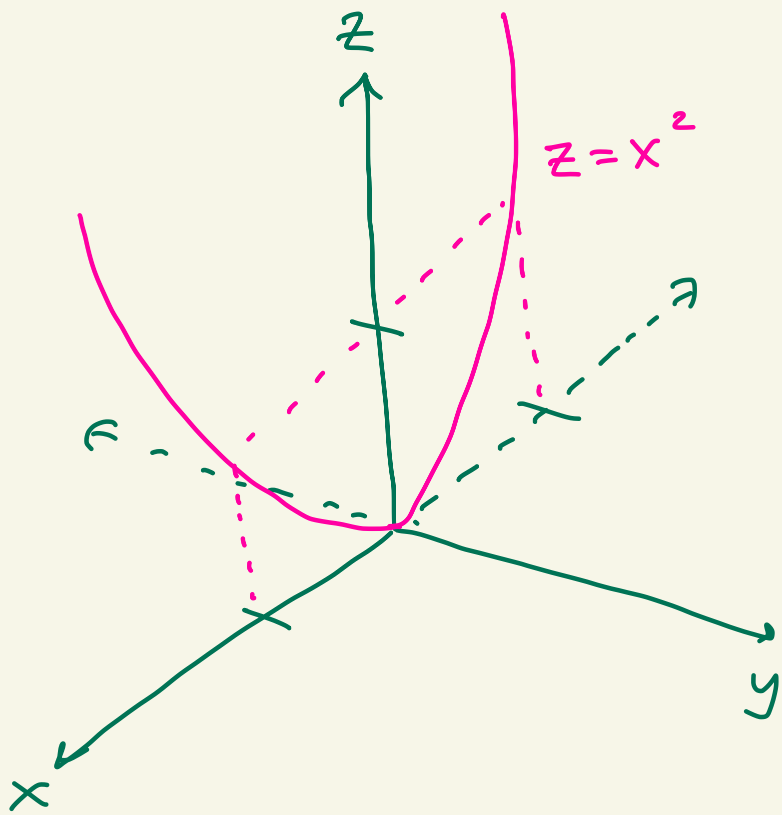


$y=0$ trace:

When $y=0$ we get
$z = x^2+y^2 = x^2+0^2 = x^2$

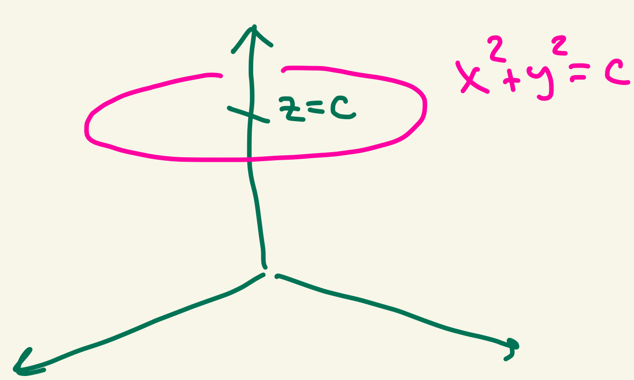
So we get $z=x^2$

The trace lives in the $y=0$ plane (the yz-axis)

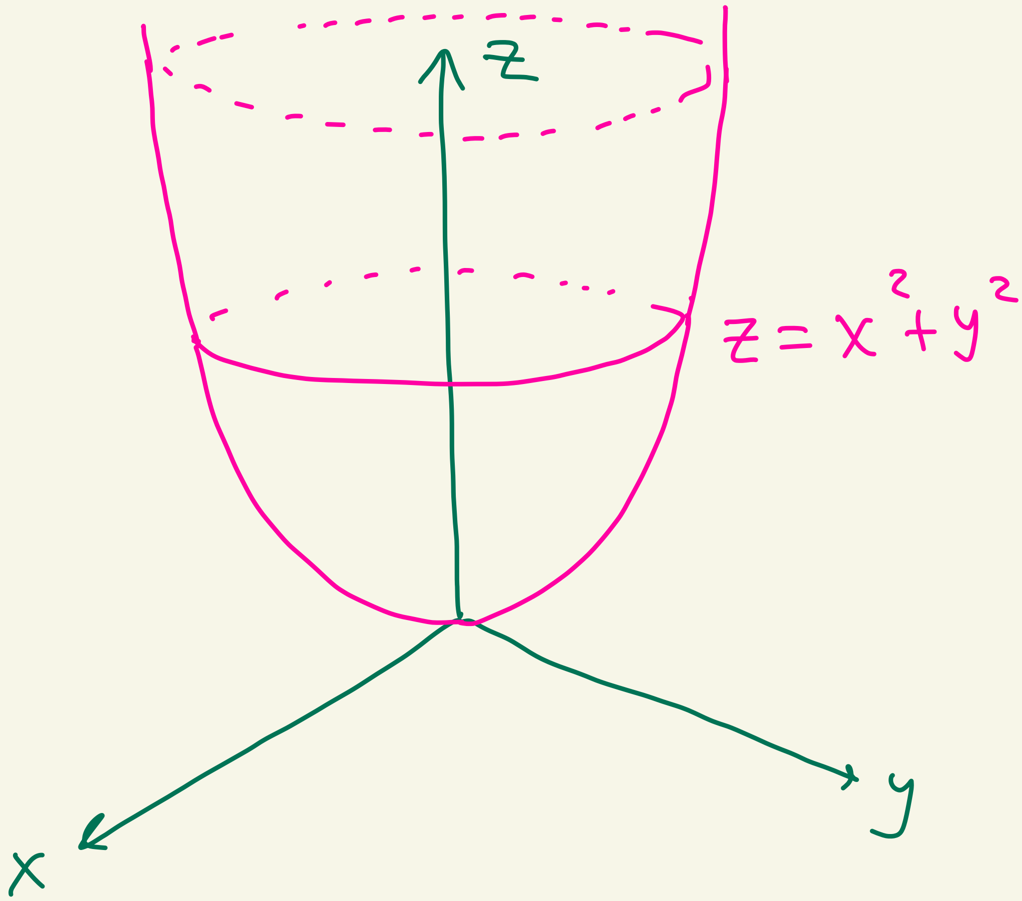


trace $z=c$:

When $z=c$ the trace is $x^2+y^2=c$ which is a circle of radius $\sqrt{c}$.

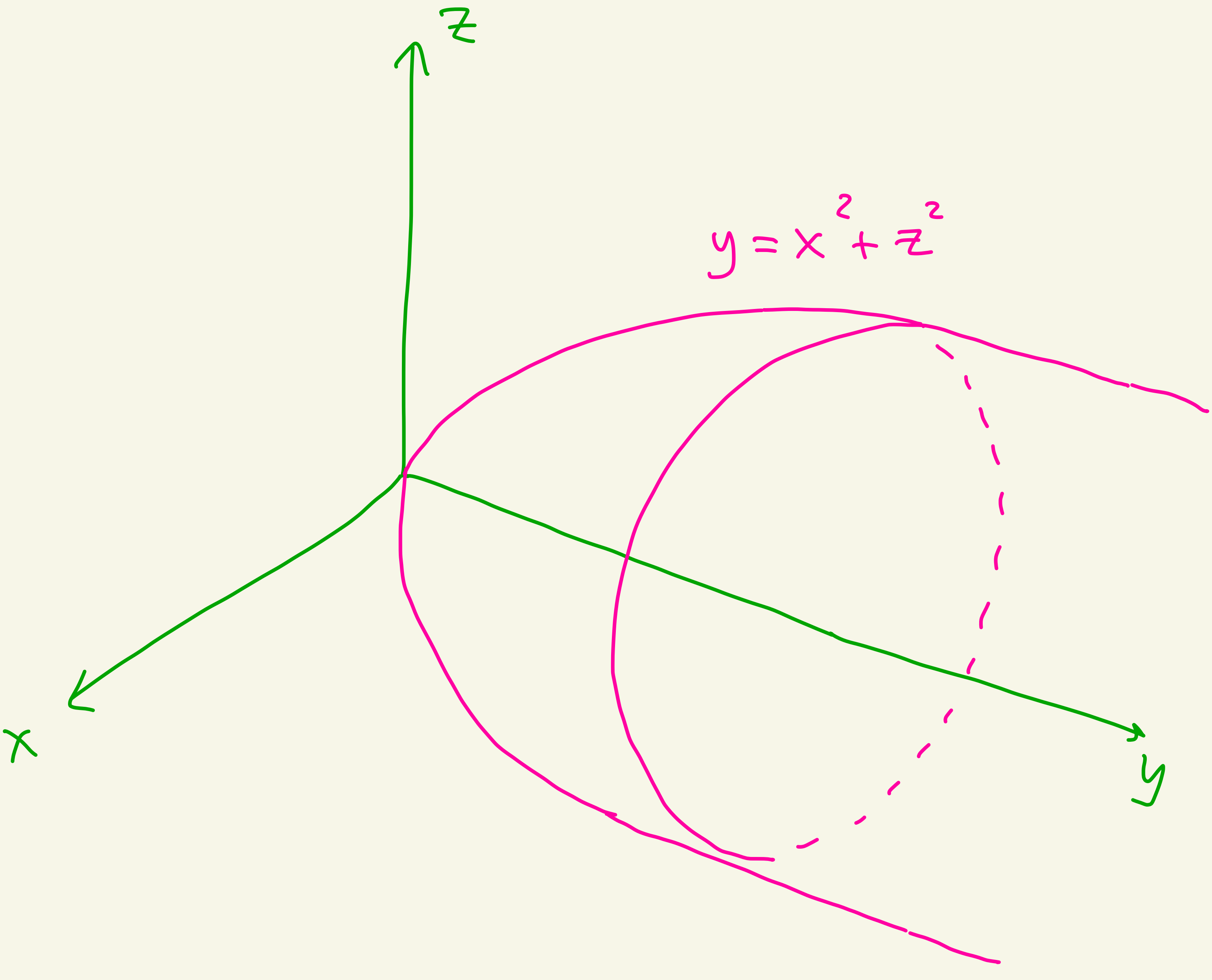


If you plot out all the traces then you get the graph of the surface $z = x^2 + y^2$.



Ex: Graph $y = x^2 + z^2$.

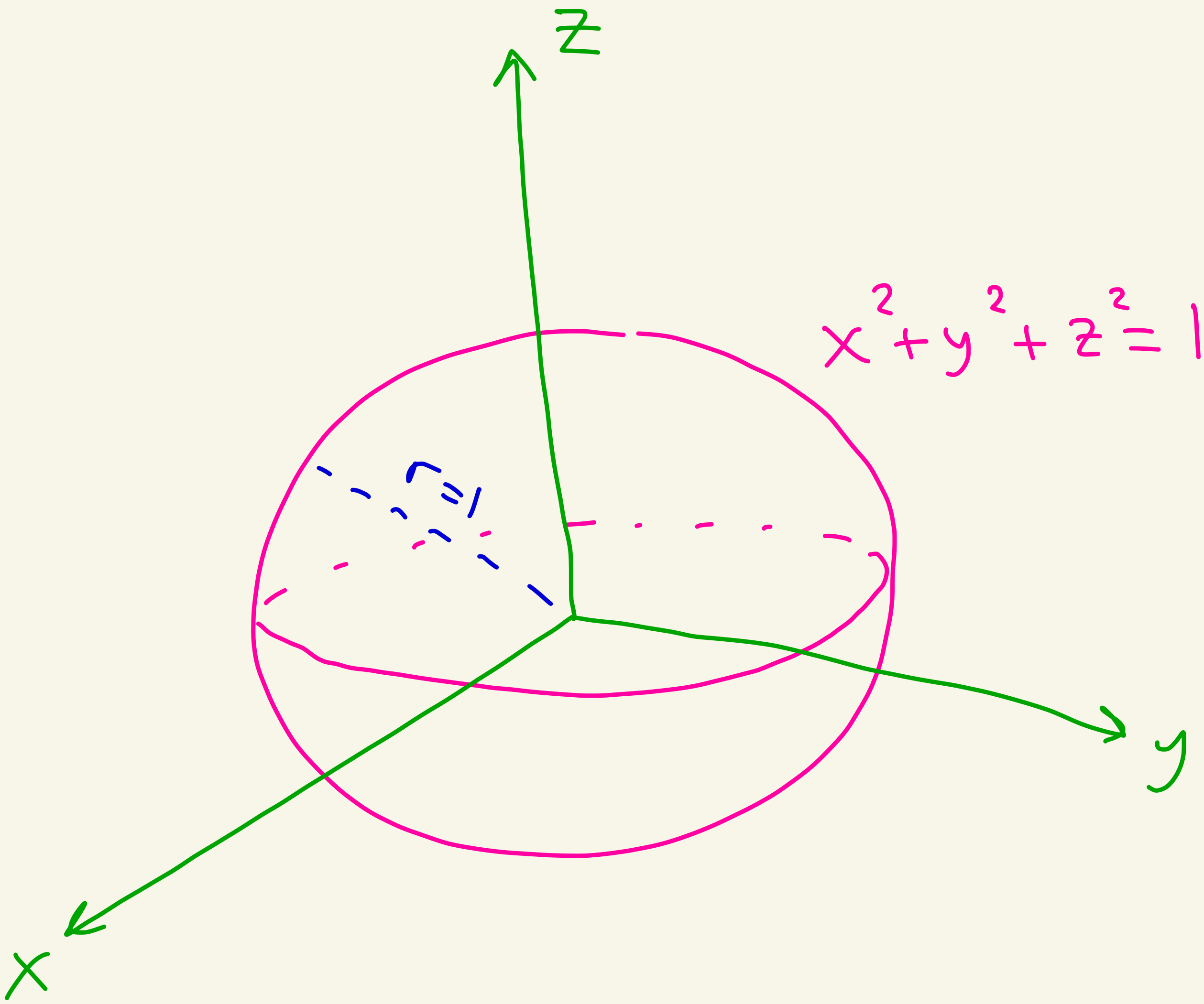
This is the same as the previous example except you graph it along the positive y-axis.



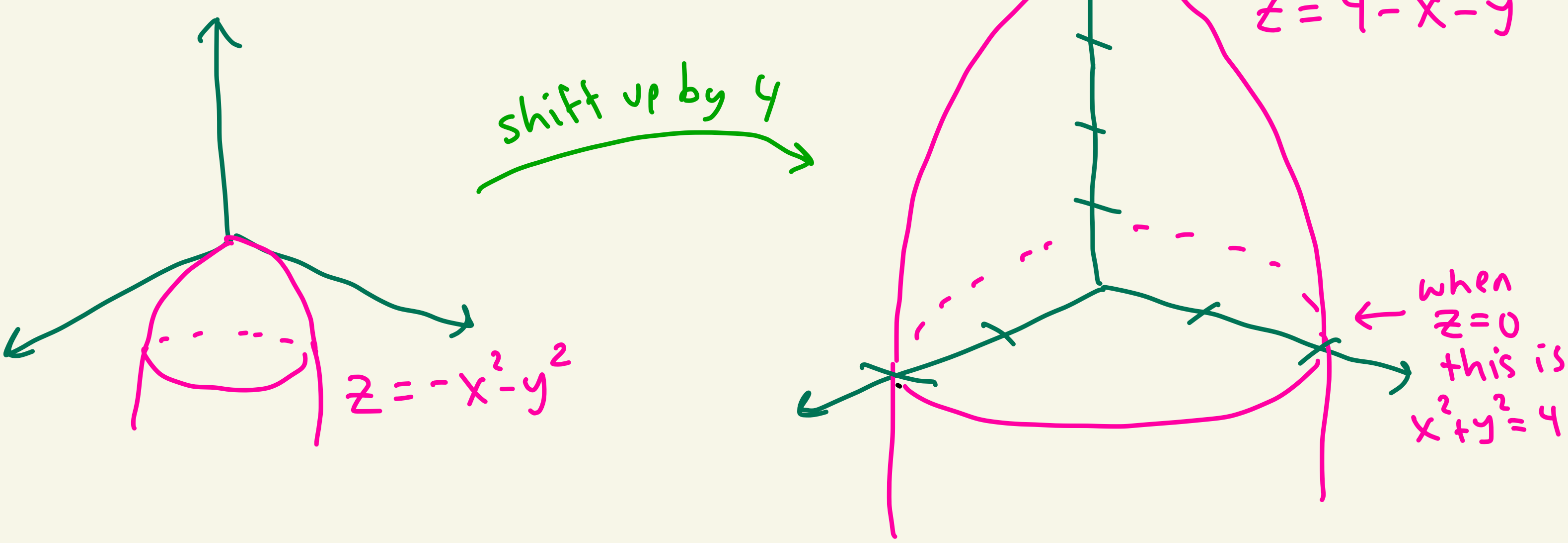
Ex:

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

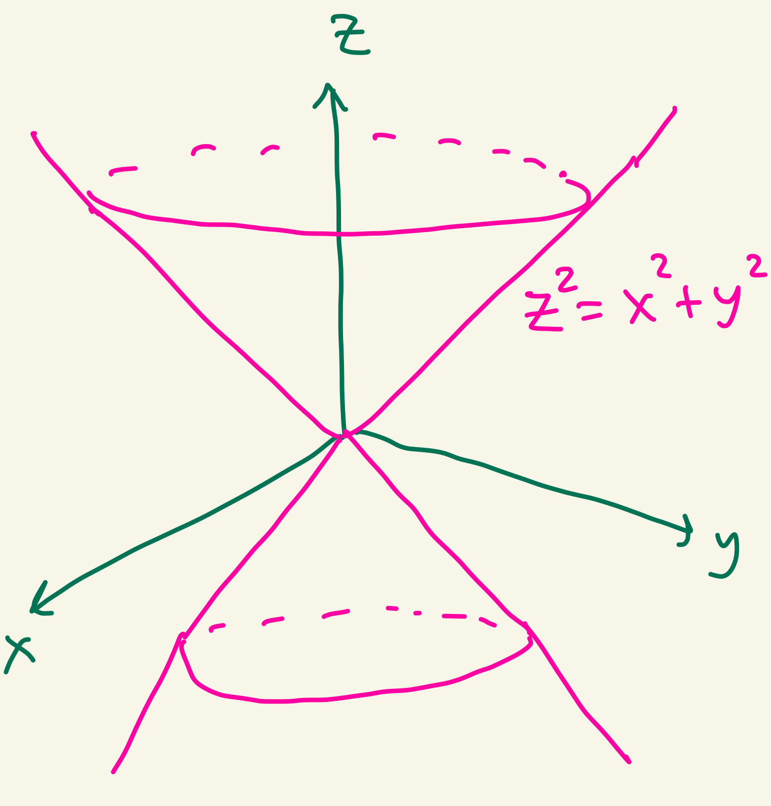
is the sphere of radius r centered at (x_0, y_0, z_0)

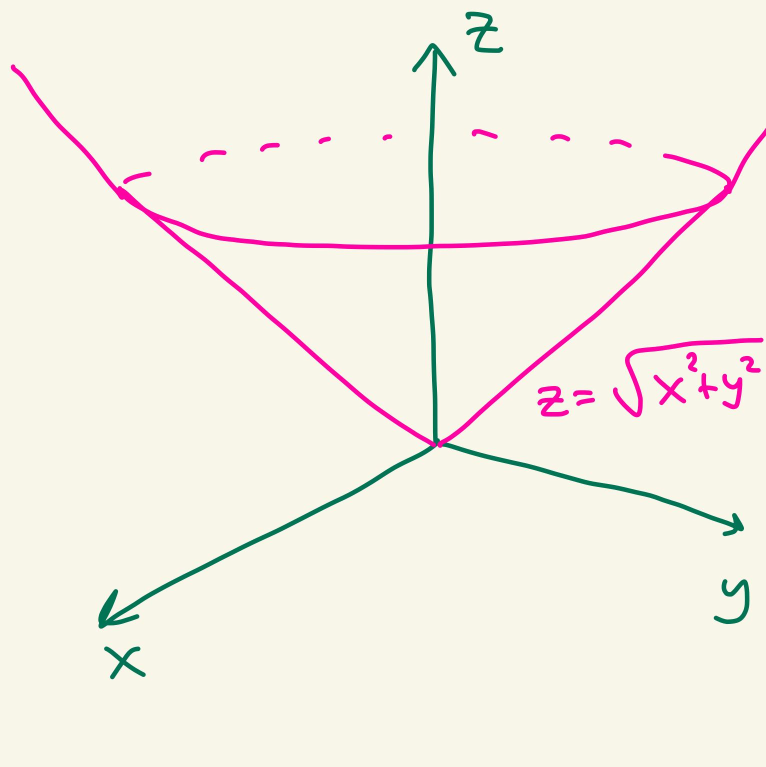


Ex: Graph $z = 4 - x^2 - y^2$



Ex: (cone)

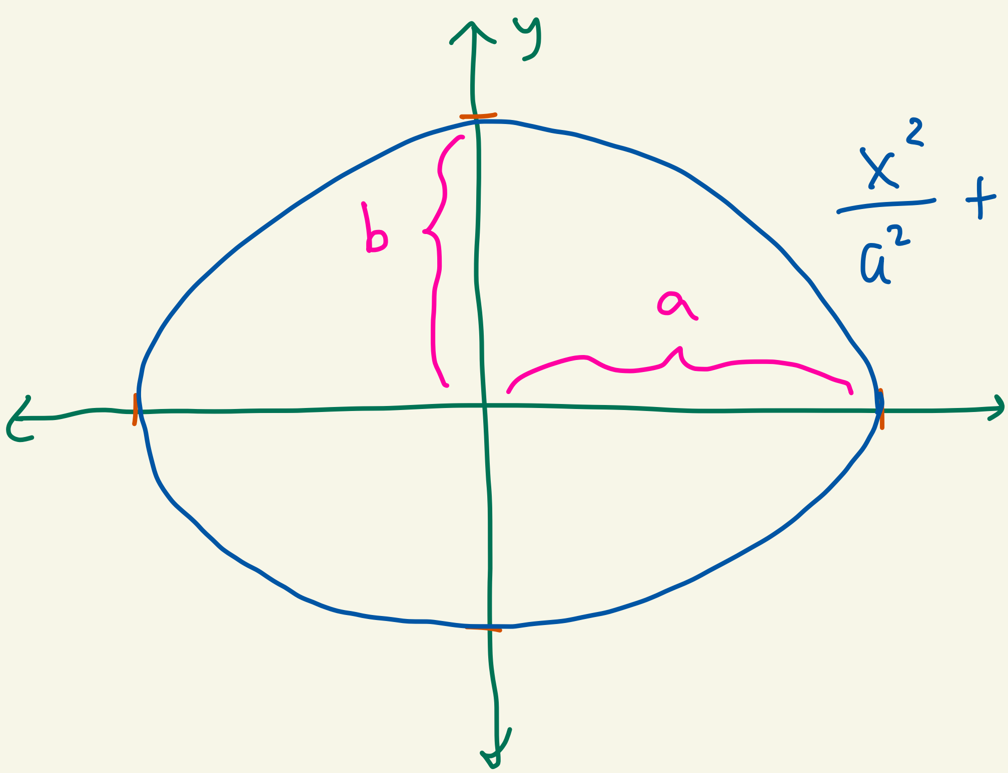




Graphing:

You can also get a 2-dimensional graph for $f(x,y)$ by plotting the various level curves $f(x,y)=z$ for different z values. The plot is in the xy-plane. It is like how mountains and valleys are drawn on a map.

The next example uses the formula for an ellipse.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Ex: Let's graph some level curves for $f(x,y) = 4x^2 + y^2$.

$z=0$: $\boxed{0 = 4x^2 + y^2}$

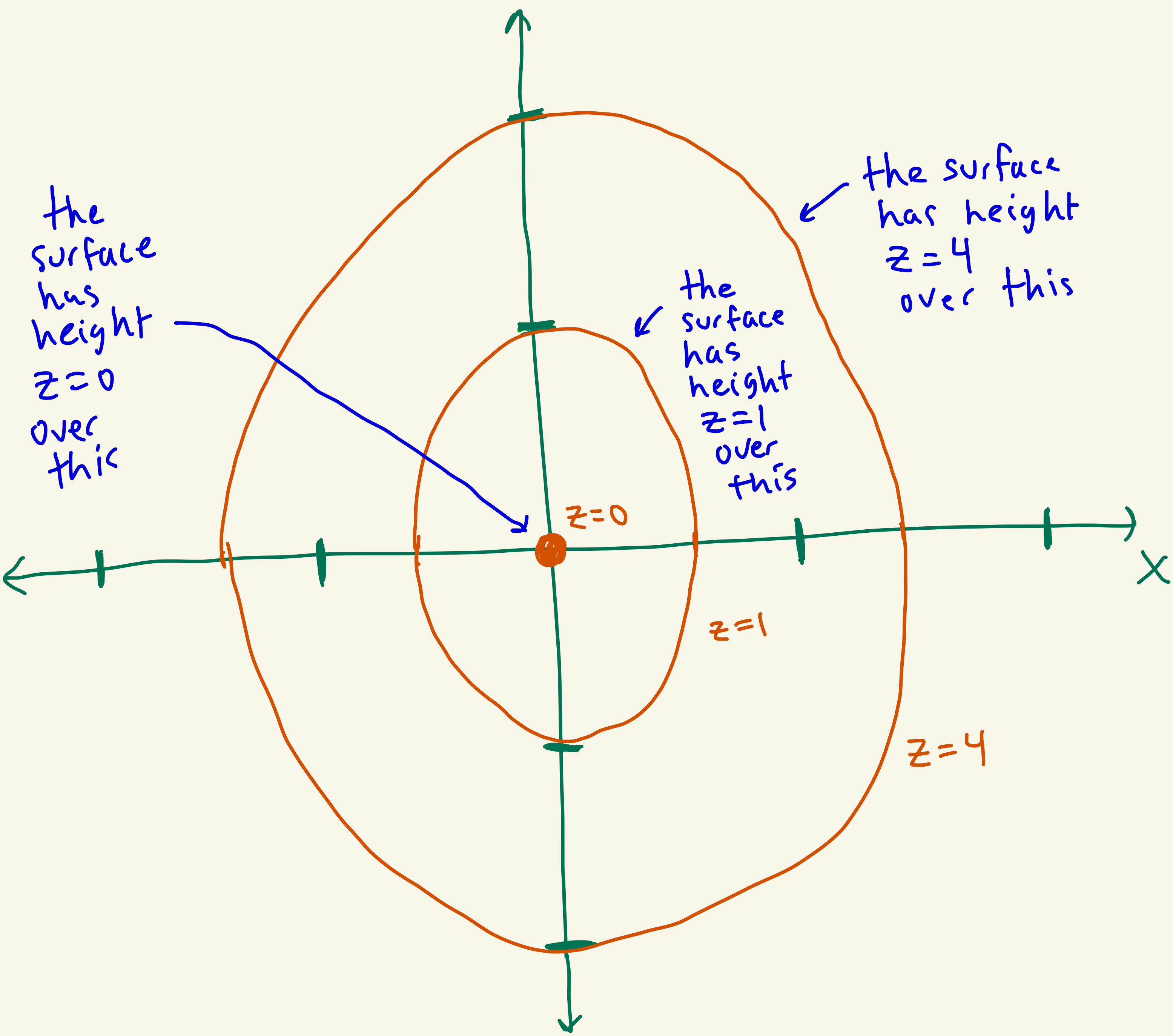
$z=1$: $1 = 4x^2 + y^2$

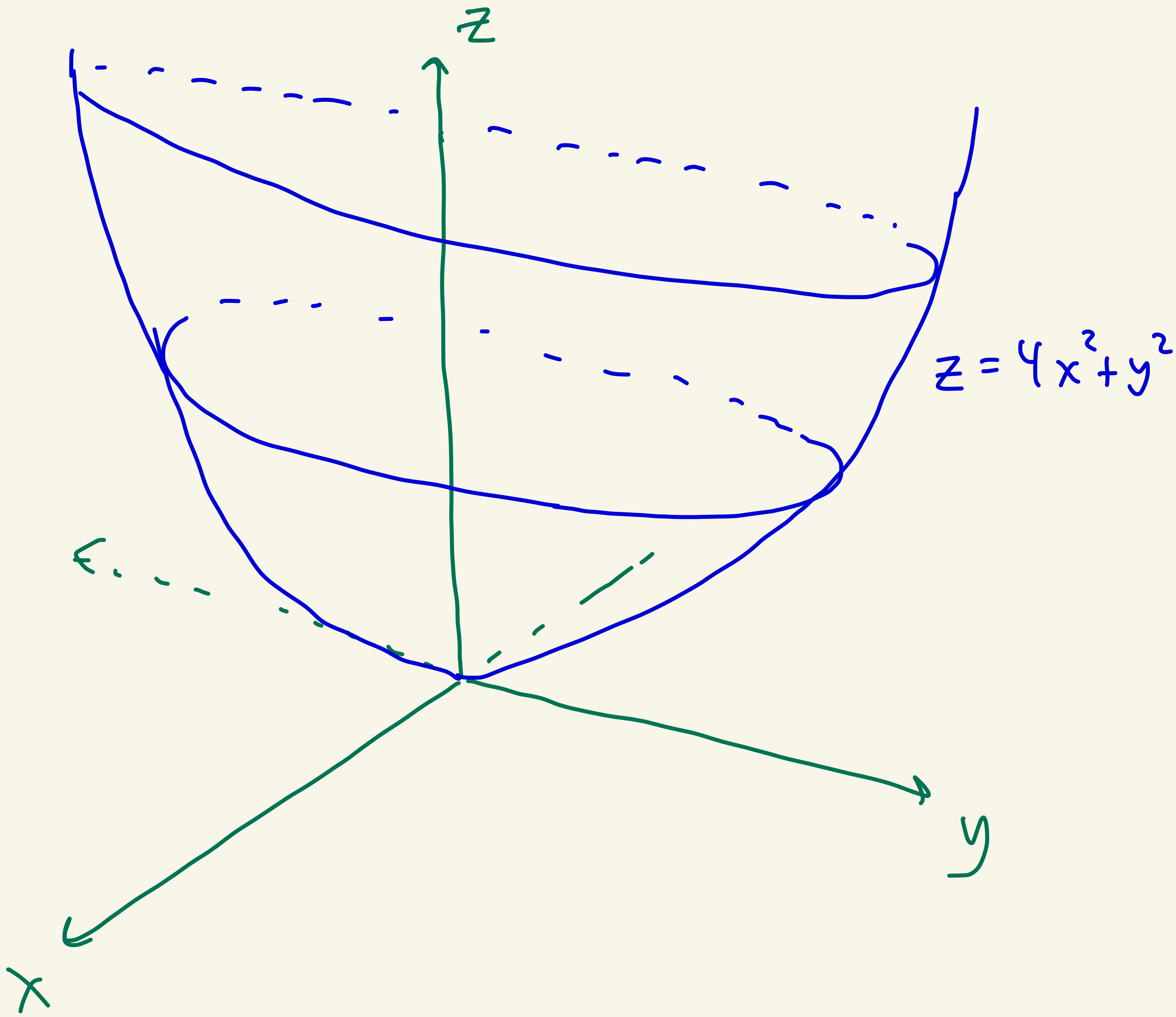
$$\boxed{1 = \frac{x^2}{\left(\frac{1}{2}\right)^2} + \frac{y^2}{1^2}}$$

$z=4$: $4 = 2x^2 + y^2$

$$\boxed{1 = \frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{2^2}}$$

$\sqrt{2} \approx 1.414$

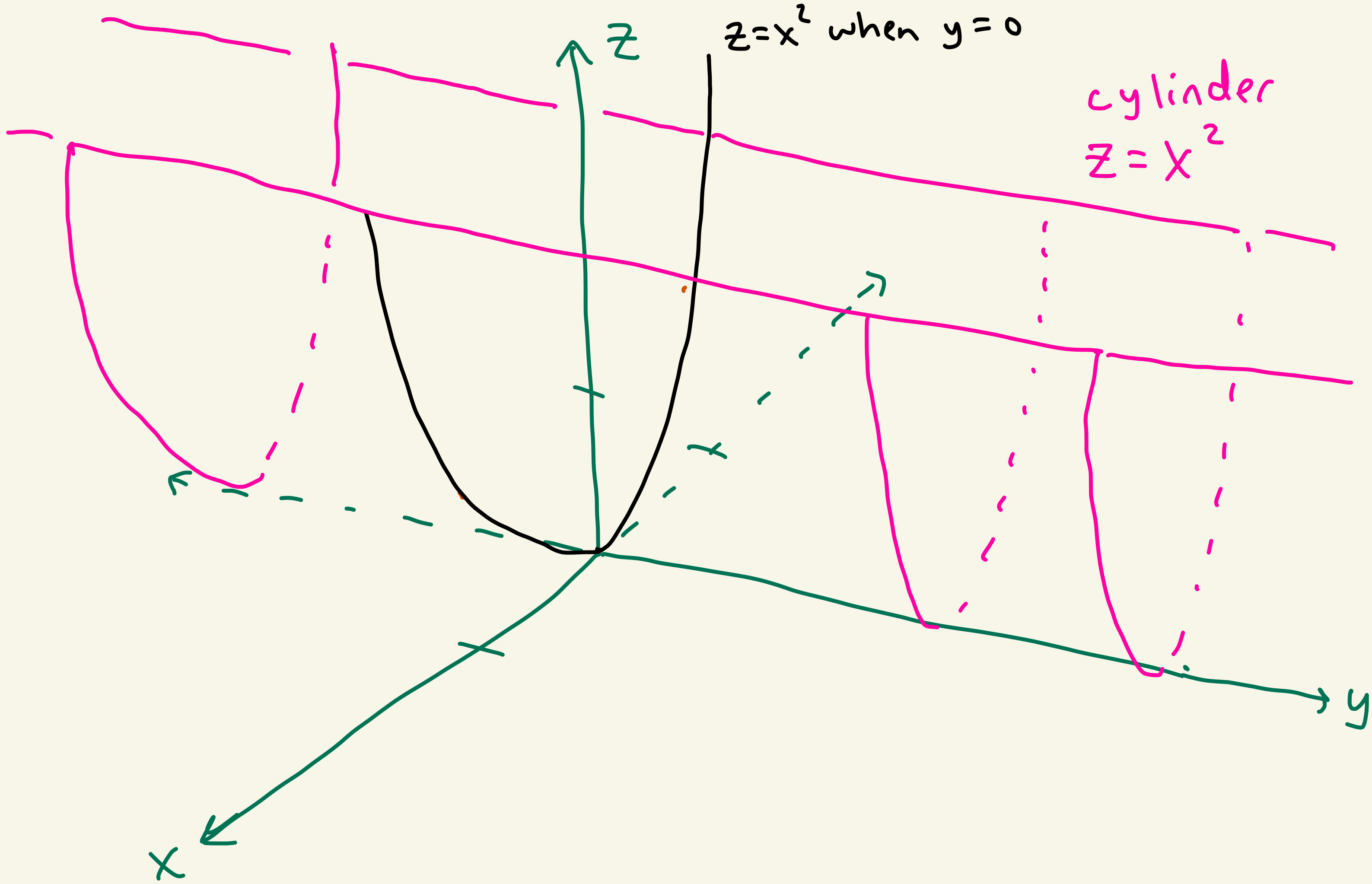




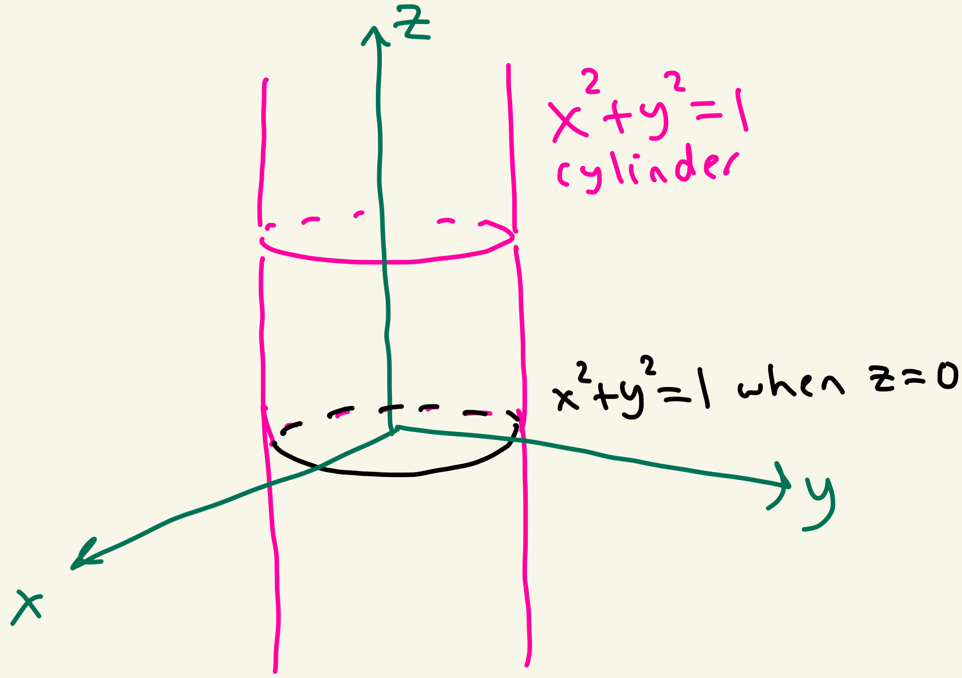
This is called an elliptic paraboloid.

Sometimes you get an equation that is missing a variable. You just draw the shape and "extend" it along the axis of the missing variable. These types of graphs are called <u>cylinders</u>, even though their shape isn't a traditional cylinder.

<u>Ex:</u> Graph the cylinder $z = x^2$



Ex: Graph the cylinder $x^2+y^2=1$



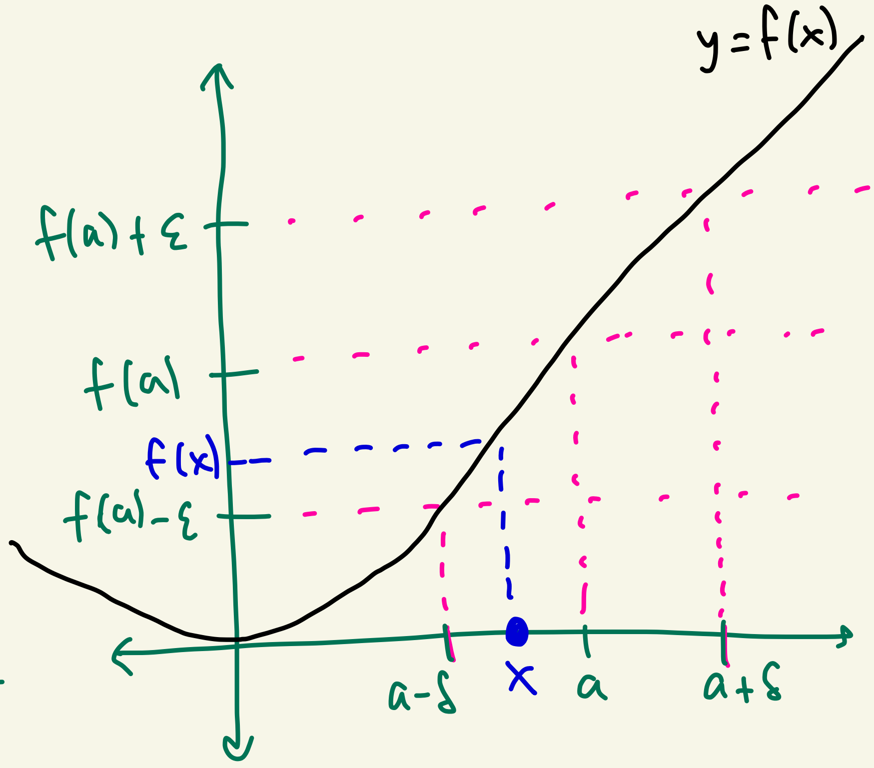
<u>Recall</u>: In Calc I, we said that $f(x)$ is continuous at $x=a$ if $\lim_{x\to a} f(x) = f(a)$

That is, if $f(x)$ tends to $f(a)$ as x tends to a from either direction.

This makes it so there are no jumps or holes in the graph.

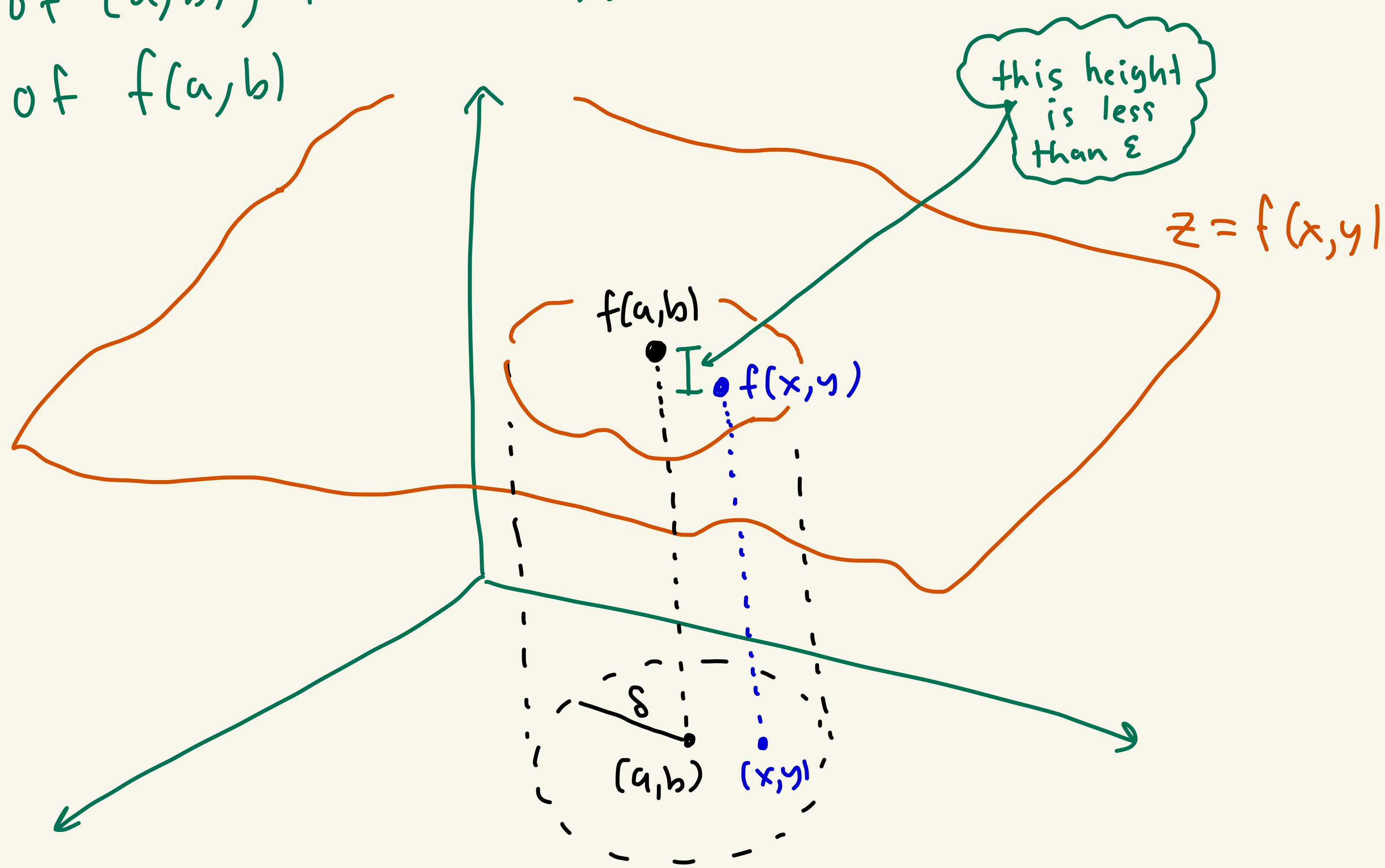
The formalism is as follows:

$f(x)$ is <u>continuous</u> at a if given any $\varepsilon > 0$ there must exist $\delta > 0$ where if x is within δ-distance of a, then $f(x)$ is within ε-distance of $f(a)$.



Def: We say that $f(x,y)$ is continuous at (a,b) if $f(x,y) \to f(a,b)$ if $(x,y) \to (a,b)$.

The formalism is:
Given any $\varepsilon > 0$ there must exist $\delta > 0$ where if (x,y) is within δ-distance of (a,b), then $f(x,y)$ is with ε-distance of $f(a,b)$



Note: Any combination of polynomials, root, exponential, trigonometric, logarithmic functions will be continuous on the fractions domain.

Ex:

$$f(x,y) = x^2 + y^2$$
$$g(x,y) = 2x\cos(xy^2) + y$$
$$h(x,y) = e^{x^2+y^2}$$

Continuous for all (x,y)

Ex: $f(x,y) = e^{\frac{1}{x} + x^2 y}$

The domain of f is all (x,y) where $x \neq 0$.
This is where f is continuous.